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The Multiplication and Division of Simple Continued Fractions

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Abstract

The finite $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}} = [a_0;a_1,\ldots,a_n]$ and infinite $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$ simple continued fractions for (rational and irrational) are considered. The *multiplication* of two simple continued fractions $[a_0;a_1,\ldots,a_n]$ and $[b_0;b_1,\ldots,b_n]$ for $(m = n,\ m \neq n)$ are discovered. Also, the *multiplicative inverse* and the *division* of the simple continued fractions are showed. The most important that we will do in this paper, we exploring how the simple continued fractions can be used to calculate the numbers $\sqrt{a}\cdot\sqrt{b}$, $\sqrt{a} \mp \sqrt{b}$ and $\frac{a}{b} \mp \sqrt{c}$. Many definitions and examples that we used of these low are presented.

Key Words: Simple continued fractions. Multiplicative. Multiplicative inverse, Division.

Introduction

There are many applications of continued fractions: combine continued fractions with the concepts of golden ratio and Fibonacci numbers, Pell equations and calculation of fundamental units in quadratic fields, reduction of quadratic forms and calculation of class numbers of imaginary quadratic field [7]. There is a pleasant connection between Chebyshev polynomials, the Pell equation and continued fractions, the latter two being understood to take place in real quadratic function fields rather than the classical case of real quadratic number fields [1]. The simple continued fractions have been studied in mathematical (Diophantine Equation, congruence ax≡b (mod m) and Pell's equations) and physical (gear ratio) [4]. The analytic of continued fractions for the real and complex values have been studied in [3,9]. However, [2,4,6] studied the continued fractions for the integer values. There are many applications of simple continued fractions (Gosper's batting average problem [8], Cryptography …). In [5,6] we show that, any number, rational or irrational, can be expression as a finite or infinite continued fraction. Also, we can solve any Diophantine Equation or congruence ax≡b (mod m). The most important, that we did in [5,6], we define the addition and subtraction of the simple continued fractions. Also, we showed that, how can we know which simple continued fraction is greater than of the other. In this paper, we will define the multiplication and division of the simple continued fractions and we will be exploring how continued fractions can be used to multiply the numbers √a·√b . We start with some definitions and theorems that we used to define the multiplication of two simple continued fractions.

Definitions

- An expression of the form $a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots}}}$ is called a continued fraction. $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$

  can be real or complex, and their numbers can be either finite or infinite.

- A continued fraction (The above fraction) is called a finite simple continued fraction if $b_n = 1$ for all n, that is

  $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$ where a_n is positive integer for all $n \geq 1$, a_0 can be any integer number. This fraction

  is sometime represented by $[a_0; a_1, a_2, \ldots, a_n]$ for finite simple continued fraction and $[a_0; a_1, a_2, \ldots]$ for infinite simple continued fraction. In this paper we will use the symbol (S.C.F.) for the simple continued fraction.

Theorem 1

A number is rational if and only if it can be expressed as a finite S.C.F. [4].

For example

$$\frac{20}{3} = 6 + \frac{2}{3} = 6 + \cfrac{1}{\cfrac{3}{2}} = 6 + \cfrac{1}{1 + \cfrac{1}{2}} = [6;1,2].$$

Remark 1

(i) If $a > b > 0$ and $\frac{a}{b} = [a_0; a_1, a_2, \ldots, a_n]$ then $\frac{b}{a} = [0; a_0, a_1, a_2, \ldots, a_n]$.

(ii) To write $-\frac{b}{a}$ $(a, b > 0)$ as S.C.F. we take the greatest integer number $\left\lfloor -\frac{b}{a} \right\rfloor$ for the first term of S.C.F.

that is $\left\lfloor -\frac{b}{a} \right\rfloor = -a'_0$ where $-a'_0 \leq -\frac{b}{a} < -a'_0 + 1$ then $-\frac{b}{a} = -a'_0 + \cfrac{1}{\cfrac{b'}{a'}}$ and we use the same

techniques as in theorem 1 to get the remaining terms for $\frac{b'}{a'}$. That is, if $\frac{b'}{a'} = [a'_1; a'_2, \ldots, a'_n]$ then

$-\frac{b}{a} = [-a'_0; a'_1, \ldots, a'_n]$.

Example 1

(1) $\frac{3}{7} = \; = [0;2,3]$ (2) $-\frac{27}{5} = \; = [-6;1,1,2]$.

Definition 1

The S.C.F. $[a_0;a_1,...,a_n]$ can be defined by

$$[a_0;a_1,...,a_n] = a_0 + \frac{K_{n-1}(a_2)}{K_n(a_1)}, \text{ or } \quad [a_0;a_1,...,a_n] = \frac{K_{n+1}(a_0)}{K_n(a_1)}.$$

where

$$\begin{array}{ll}
K_0(a_0) = 1 & K_0(a_1) = 1 \\
K_1(a_0) = a_0 & K_1(a_1) = a_1 \\
K_2(a_0) = a_0a_1 + 1 & K_2(a_1) = a_1\ a_2 + 1 \\
K_3(a_0) = a_0\ a_1a_2 + a_0 + a_2 & K_3(a_1) = a_1\ a_2a_3 + a_1 + a_3 \\
\vdots & \vdots \\
K_i(a_0) = a_{i-1}\ K_{i-1}(a_0) + K_{i-2}(a_0) & K_i(a_1) = a_i\ K_{i-1}(a_1) + K_{i-2}(a_1)
\end{array}$$

In general

$$K_i(a_j) = a_{i+(j-1)}\ K_{i-1}(a_j) + K_{i-2}(a_j) \quad , \qquad i = 1,2,...,n \quad , \quad j = 0,1,...,n$$

$$K_{-i}(a_j) = 0 \qquad\qquad K_0(a_j) = 1.$$ [5,6]

Example 2

Evaluate the S.C.F. [2;2,2].

Solution

Let $[2;2,2] = [a_0;a_1,a_2]$, then $n = 2$. From definition 1 we get:

$$[a_0;a_1,a_2] = a_0 + \frac{K_1(a_2)}{K_2(a_1)} = a_0 + \frac{a_2}{a_1a_2 + 1},$$

therefore

$$[2;2,2]=2+\frac{2}{(2\cdot 2+1)}=2+\frac{2}{5}=\frac{12}{5}.$$

Lemma

1) $[a_0;a_1,\dots,a_n]=[a_0;a_1,\dots,a_n-1,1]$.

2) $[c_0;c_1,\dots,c_{j-1},0,c_{j+1},\dots,c_n]=[c_0;c_1,\dots,c_{j-1}+c_{j+1},\dots,c_n]$.

3) $[c_0;c_1,\dots,c_{j-1},0,0,c_{j+2},\dots,c_n]=[c_0;c_1,\dots,c_{j-1},c_{j+2},\dots,c_n]$.

Definition 2

Let $[a_0;a_1,\dots,a_m]$ and $[b_0;b_1,\dots,b_n]$ be two S.C.F., a_0,b_0 are non-negative, we define their *multiplication* by:

(1) If $m=n$ then

$$[a_0;a_1,\dots,a_n]\times[b_0;b_1,\dots,b_n]=[d_0;d_1,\dots,d_n] \tag{2a}$$

where

$$d_0=a_0b_0$$

$$d_1=\left\lfloor\frac{a_1b_1}{a_0a_1+b_0b_1+1}\right\rfloor$$

$$d_2=\left\lfloor\frac{a_0b_2(a_1a_2+1)+a_2b_0(b_1b_2+1)+a_2b_2}{(a_1a_2+1)(b_1b_2+1)-d_1[a_0b_2(a_1a_2+1)+a_2b_0(b_1b_2+1)+a_2b_2]}\right\rfloor$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$d_i=\left\lfloor\frac{K_i(a_1)K_i(b_1)K_{i-3}(d_2)-[a_0K_i(a_1)K_{i-1}(b_2)+b_0K_{i-1}(a_2)K_i(b_1)+K_{i-1}(a_2)K_{i-1}(b_2)]K_{i-2}(d_1)}{[a_0K_i(a_1)K_{i-1}(b_2)+b_0K_i(b_1)K_{i-1}(a_2)+K_{i-1}(a_2)K_{i-1}(b_2)]K_{i-1}(d_1)-K_i(a_1)K_i(b_1)K_{i-2}(d_2)}\right\rfloor \quad \text{If } i \text{ odd,}$$

$$d_i=\left\lfloor\frac{[a_0K_{i-1}(b_2)K_i(a_1)+b_0K_{i-1}(a_2)K_i(b_1)+K_{i-1}(a_2)K_{i-1}(b_2)]K_{i-2}(d_1)-K_i(a_1)K_i(b_1)K_{i-3}(d_2)}{K_i(a_1)K_i(b_1)K_{i-2}(d_2)-[a_0K_i(a_1)K_{i-1}(b_2)+b_0K_{i-1}(a_2)K_i(b_1)+K_{i-1}(a_2)K_{i-1}(b_2)]K_{i-1}(d_1)}\right\rfloor \quad \text{If } i \text{ even}$$

for $i=2,3,\dots,n$.

The last term d_n of the resulting S.C.F. is to be expanded again as a S.C.F. if necessary and not to be treated as the greatest integer number as the preceding terms have been treated.

(2) If $m \neq n$ (suppose that $m < n$) then

$$[a_0;a_1,...,a_m]\times[b_0;b_1,...,b_m,b_{m+1},...,b_n]=[d'_0;d'_1,...,d'_m,d'_{m+1},...,d'_n] \quad (2b)$$

where $d'_j = d_j$ *for* $j = 1,2,...,m,$

and d_j *as we did for case m = n while* $d'_j = d_{j,j-m}$

$$d_{j,j-m} = \left[\frac{K_m(a_1)K_j(b_1)K_{j-3}(d_2)-[a_0K_m(a_1)K_{j-1}(b_2)+b_0K_j(b_1)K_{m-1}(a_2)+K_{m-1}(a_2)K_{j-1}(b_2)]K_{j-2}(d_1)}{[a_0K_m(a_1)K_{j-1}(b_2)+b_0K_j(b_1)K_{m-1}(a_2)+K_{m-1}(a_2)K_{j-1}(b_2)]K_{j-1}(d_1)-K_m(a_1)K_j(b_1)K_{j-2}(d_2)}\right] \text{ if } j \text{ is odd,}$$

$$d_{j,j-m} = \left[\frac{[a_0K_m(a_1)K_{j-1}(b_2)+b_0K_j(b_1)K_{m-1}(a_2)+K_{m-1}(a_2)K_{j-1}(b_2)]K_{j-2}(d_1)-K_m(a_1)K_j(b_1)K_{j-3}(d_2)}{K_m(a_1)K_j(b_1)K_{j-2}(d_2)-[a_0K_m(a_1)K_{j-1}(b_2)+b_0K_j(b_1)K_{m-1}(a_2)+K_{m-1}(a_2)K_{j-1}(b_2)]K_{j-1}(d_1)}\right] \text{ if } j \text{ is even.}$$

For $j = m+1, m+2, \ldots, n$.

Also, the last term is to be treated as a simple continued fraction.

Example 3

Find $[1;2]\times[1;3]$.

Solution:

Let $[1;2] = [a_0;a_1]$ and $[1;3] = [b_0;b_1]$, we get $m = n = 1$.

From Equation (2a) we have

$[1;2]\times[1;3] = [a_0;a_1]\times[b_0;b_1] = [d_0;d_1]$,

where d_1 is the last term and

$$d_0 = a_0b_0 = 1\cdot 1 = 1$$

$$d_1 = \frac{a_1b_1}{a_0a_1+b_0b_1+1} = \frac{2\cdot 3}{1\cdot 2+1\cdot 3+1} = \frac{6}{6} = 1$$

therefore

$[1;2]\times[1;3] = [1;1]$

$= 2$ (Lemma 1)

To check, we have $[1;2]\times[1;3] = \frac{3}{2}\times\frac{4}{3} = 2$

Example 4

Find [2;1,4]×[0;2,3].

Solution:

Let [2;1,4]=[$a_0;a_1,a_2$] and [0;2,3]=[$b_0;b_1,b_2$], *m*=*n*=2. From Equation (2a), we have

[2;1,4]×[0;2,3] = [$a_0;a_1,a_2$]×[$b_0;b_1,b_2$] = [$d_0;d_1,d_2$] ,

where d_2 is the last term and

$$d_0 = a_0 b_0 = 2\cdot 0 = 0$$

$$d_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1} \right\rfloor = \left\lfloor \frac{1\cdot 2}{2\cdot 1 + 0\cdot 2 + 1} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0$$

$$d_2 = \frac{a_0 b_2 (a_1 a_2 + 1) + a_2 b_0 (b_1 b_2 + 1) + a_2 b_2}{(a_1 a_2 + 1)(b_1 b_2 + 1) - d_1 [a_0 b_2 (a_1 a_2 + 1) + a_2 b_0 (b_1 b_2 + 1) + a_2 b_2]}$$

$$= \frac{2\cdot 3\cdot (1\cdot 4 + 1) + 4\cdot 0(2\cdot 3 + 1) + 4\cdot 3}{(1\cdot 4 + 1)(2\cdot 3 + 1) - 0\cdot [2\cdot 3\cdot (1\cdot 4 + 1) + 4\cdot 0(2\cdot 3 + 1) + 4\cdot 3]}$$

$$= \frac{30 + 12}{5\cdot 7} = \frac{42}{35} = \frac{6}{5} = [1;5]$$

therefore

[2;1,4]×[0;2,3] = [0;0,1,5]

= [1;5] (Lemma 3)

To check, we have $[2;1,4]\times[0;2,3] = \frac{14}{5}\times\frac{3}{7} = \frac{6}{5}$

and $[1;5] = 1 + \frac{1}{5} = \frac{5\cdot 1 + 1}{5} = \frac{5+1}{5} = \frac{6}{5}$

Example 5

Find $[2;1,4]\times[1;2]$.

Solution:

Let [1;2]=[$a_0;a_1$] and [2;1,4]=[$b_0;b_1,b_2$], *m*=1, *n*=2, *m*≠*n*, from Equation (2b), we get

$[2;1,4]\times[1;2] = [a_0;a_1]\times[b_0;b_1,b_2] = [d_0;d_1,d_2']$

where d_2' is the last term and

$$d_0 = d_0' = a_0 b_0 = 2\cdot 1 = 2$$

$$d_1 = d_1' = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1} \right\rfloor = \left\lfloor \frac{2\cdot 1}{1\cdot 2 + 2\cdot 1 + 1} \right\rfloor = \left\lfloor \frac{2}{5} \right\rfloor = 0$$

$$d_2' = \frac{a_0 b_2 a_1 + b_0(b_1 b_2 + 1) + b_2}{a_1\left(b_1 b_2 + 1\right) - d_1'[a_0 b_2 a_1 + b_0(b_1 b_2 + 1) + b_2]} = \frac{1\cdot 4\cdot 2 + 2\cdot(1\cdot 4 + 1) + 4}{2\cdot(1\cdot 4 + 1) - 0}$$

$$= \frac{8+10+4}{10} = \frac{22}{10} = \frac{11}{5}$$

= [2;5]

therefore

$[2;1,4]\times[1;2]$ = [2;0,2,5]

= [4;5] (Lemma 2)

To check, we have $[2;1,4]\times[1;2] = \frac{14}{5}\times\frac{3}{2} = \frac{21}{5}$

and $[4;5] = 4 + \frac{1}{5} = \frac{5\cdot 4+1}{5} = \frac{20+1}{5} = \frac{21}{5}$

Example 6

Find $[1;4]\times[2;3,1,2,4]$.

Solution:

Let [1;4]=$[a_0;a_1]$ and [2;3,1,2,4]=$[b_0;b_1,b_2,b_3,b_4]$ we get *m*=1, *n*=4, *m*≠*n*, from Equation (2b) we have

$[1;4]\times[2;3,1,2,4] = [a_0;a_1]\times[b_0;b_1,b_2,b_3,b_4] = [d_0;d_1,d_2',d_3',d_4']$,

where d_4' is the last term and

$$d_0 = d_0' = a_0 b_0 = 1\cdot 2 = 2$$

$$d_1 = d_1' = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1} \right\rfloor = \left\lfloor \frac{4\cdot 3}{1\cdot 4 + 2\cdot 3 + 1} \right\rfloor = 1$$

$$d_2' = \left\lfloor \frac{a_0 b_2 a_1 + b_0 (b_1 b_2 + 1) + b_2}{a_1 \left(b_1 b_2 + 1\right) - d_1' [a_0 b_2 a_1 + b_0 (b_1 b_2 + 1) + b_2]} \right\rfloor$$

$$= \left\lfloor \frac{1\cdot 4\cdot 1 + 2\cdot(3\cdot 1 + 1) + 1}{4\cdot 4 - 13\cdot 1} \right\rfloor = \left\lfloor \frac{4+8+1}{16-13} \right\rfloor = \left\lfloor \frac{13}{3} \right\rfloor = 4$$

$$d_3' = \left\lfloor \frac{a_1 K_3(b_1) - [a_0 a_1 K_2(b_2) + b_0 K_3(b_1) + K_2(b_2)] d_1'}{[a_0 a_1 K_2(b_2) + b_0 K_3(b_1) + K_2(b_2)] K_2(d_1') - a_1 K_3(b_1) d_2'} \right\rfloor$$

$$= \left\lfloor \frac{4(3\cdot 1\cdot 2 + 3 + 2) - [4\cdot 3 + 2\cdot 11 + 3]}{37(5) - 44(4)} \right\rfloor = \left\lfloor \frac{44-37}{185-176} \right\rfloor = \left\lfloor \frac{7}{9} \right\rfloor = 0$$

$$d_4' = \frac{[a_0 a_1 K_3(b_2) + b_0 K_4(b_1) + K_3(b_2)] K_2(d_1') - a_1 K_4(b_1) d_2'}{a_1 K_4(b_1) K_2(d_2') - [a_0 a_1 K_3(b_2) + b_0 K_4(b_1) + K_3(b_2)] K_3(d_1')}$$

$$= \frac{[4\cdot 13 + 2(24+12+8+3+1) + 13](5) - 4\cdot 48\cdot 4}{192(1) - 161(1)} = \frac{161(5) - 192(4)}{192 - 161}$$

$$= \frac{37}{21} = [1;5,6]$$

therefore

$$[1;4]\times[2;3,1,2,4] = [2;1,4,0,1,5,6]$$

$$= [2;1,5,5,6] \qquad \text{(Lemma 2)}$$

To check, we have $[1;4]\times[2;3,1,2,4] = \frac{5}{4}\times\frac{109}{48} = \frac{545}{192}$

and $[2;1,5,5,6] = 2 + \cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{5+\cfrac{1}{6}}}} = 2 + \cfrac{1}{1+\cfrac{1}{5+\cfrac{6}{31}}} = 2 + \cfrac{1}{1+\cfrac{31}{161}} = \frac{545}{192}$

Definition 3

Let $[a_0;a_1,...,a_n]$ be a S.C.F., then we define the *multiplicative inverse* of $[a_0;a_1,...,a_n]$ as $\frac{1}{[a_0;a_1,...,a_n]}=[0;a_0,a_1,...,a_n]$.

Example 7

Find the multiplicative inverse of $[0;1,1,4]$.

Solution:

From definition 3 we have:

$$\frac{1}{[0;1,1,4]}=[0;0,1,1,4]$$

$$=[1;1,4]. \qquad \text{(Lemma 3)}$$

To check, we have

$$\frac{1}{[0;1,1,4]}=\frac{1}{\frac{5}{9}}=\frac{9}{5}$$

and $$[1;1,4]=1+\frac{1}{1+\frac{1}{4}}=1+\frac{4}{5}=\frac{9}{5}$$

Definition 4

Let $[a_0;a_1,...,a_m]$ and $[b_0;b_1,...,b_n]$ be two S.C.F. we define the *Division* by $[a_0;a_1,...,a_m]\div[b_0;b_1,...,b_n]=$ $[a_0;a_1,...,a_m]\times\frac{1}{[b_0;b_1,...,b_n]}$ where $\frac{1}{[b_0;b_1,...,b_n]}=[0;b_0,b_1,...,b_n]$

Example 8

Find $[35;1,2,2]\div[3;1,1,3]$.

Solution:

Let $[35;1,2,2] = [a_0;a_1,a_2,a_3]$ and $\frac{1}{[3;1,1,3]} = [0;3,1,1,3] = [b_0;b_1,b_2,b_3,b_4]$, we get m=3, n=4, $m \neq n$, from Equation (2b), we have

$$[35;1,2,2] \times [0;3,1,1,3] = [a_0;a_1,a_2,a_3] \times [b_0;b_1,b_2,b_3,b_4] = [d_0;d_1,d_2,d_3,d'_4],$$

where d'_4 is the last term and

$$d_0 = d'_0 = a_0 b_0 = 35 \cdot 0 = 0$$

$$d_1 = d'_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1} \right\rfloor = \left\lfloor \frac{1 \cdot 3}{35 \cdot 1 + 0 \cdot 3 + 1} \right\rfloor = \left\lfloor \frac{3}{36} \right\rfloor = 0$$

$$\begin{aligned} d_2 = d'_2 &= \left\lfloor \frac{a_0 b_2 (a_1 a_2 + 1) + a_2 b_0 (b_1 b_2 + 1) + a_2 b_2}{(a_1 a_2 + 1)(b_1 b_2 + 1) - d'_1 [a_0 b_2 (a_1 a_2 + 1) + a_2 b_0 (b_1 b_2 + 1) + a_2 b_2]} \right\rfloor \\ &= \left\lfloor \frac{35 \cdot 3 + 2}{3 \cdot 4} \right\rfloor = \left\lfloor \frac{107}{12} \right\rfloor = 8 \end{aligned}$$

$$\begin{aligned} d_3 = d'_3 &= \left\lfloor \frac{K_3(a_1)K_3(b_1)K_0(d'_2) - [a_0 K_3(a_1)K_2(b_2) + b_0 K_3(b_1)K_2(a_2) + K_2(a_2)K_2(b_2)]K_1(d'_1)}{[a_0 K_3(a_1)K_2(b_2) + b_0 K_3(b_1)K_2(a_2) + K_2(a_2)K_2(b_2)]K_2(d'_1) - K_3(a_1)K_3(b_1)K_1(d'_2)} \right\rfloor \\ &= \left\lfloor \frac{K_3(a_1)K_3(b_1)}{[a_0 K_3(a_1)K_2(b_2) + K_2(a_2)K_2(b_2)] - K_3(a_1)K_3(b_1) \cdot d'_2} \right\rfloor = \left\lfloor \frac{7 \cdot 7}{[35 \cdot 7 \cdot 2 + 5 \cdot 2] - 7 \cdot 7 \cdot 8} \right\rfloor \\ &= \left\lfloor \frac{49}{108} \right\rfloor = 0 \end{aligned}$$

$$\begin{aligned} d'_4 &= \frac{[a_0 K_3(a_1)K_3(b_2) + K_2(a_2)K_3(b_2)] - K_3(a_1)K_4(b_1)K_1(d'_2)}{K_3(a_1)K_4(b_1)K_2(d'_2)} \\ &= \frac{[35 \cdot 7 \cdot 7 + 7 \cdot 5] - 7 \cdot (b_1 b_2 b_3 b_4 + b_1 b_2 + b_1 b_4 + b_3 b_4 + 1) \cdot 8}{7 \cdot (b_1 b_2 b_3 b_4 + b_1 b_2 + b_1 b_4 + b_3 b_4 + 1)} \\ &= \frac{35 \cdot 7 \cdot 7 + 7 \cdot 5 - 7 \cdot 25 \cdot 8}{7 \cdot 25} \\ &= \frac{350}{175} = 2 \end{aligned}$$

therefore

$$\begin{aligned} [35;1,2,2] \div [3;1,1,3] &= [35;1,2,2] \times [0;3,1,1,3] \\ &= [0;0,8,0,2] \\ &= [8;0,2] && \text{(Lemma 3)} \\ &= 10 && \text{(Lemma 2)} \end{aligned}$$

To check, we have $[35;1,2,2] \div [3;1,1,3] = \frac{250}{7} \div \frac{25}{7} = \frac{250}{7} \times \frac{7}{25} = 10$.

Example 9

Find $[1;2,2] \div [1;3]$.

Solution:

Let $[1;2,2] = [a_0;a_1,a_2]$ and $\frac{1}{[1;3]} = [0;1,3] = [b_0;b_1,b_2]$, we get $m = n = 2$,

from Equation (2a), we have

$[1;2,2] \times [0;1,3] = [a_0;a_1,a_2] \times [b_0;b_1,b_2] = [d_0;d_1,d_2]$,

where d_2 is the last term and

$$d_0 = a_0 b_0 = 1 \cdot 0 = 0$$

$$d_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1} \right\rfloor = \left\lfloor \frac{2 \cdot 1}{1 \cdot 2 + 0 \cdot 1 + 1} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0$$

$$d_2 = \frac{a_0 b_2 (a_1 a_2 + 1) + a_2 b_2}{(a_1 a_2 + 1)(b_1 b_2 + 1) - d_1 [a_0 b_2 (a_1 a_2 + 1) + a_2 b_2]} = \frac{1 \cdot 3 \cdot 5 + 2 \cdot 3}{5 \cdot 4}$$

$$= \frac{21}{20}$$

$$= [1;20]$$

therefore

$$[1;2,2] \div [1;3] = [1;2,2] \times [0;1,3]$$

$$= [0;0,1,20]$$

$$= [1;20] \qquad \text{(Lemma 3)}$$

To check, we have $[1;2,2] \div [1;3] = \frac{7}{5} \div \frac{4}{3} = \frac{7}{5} \times \frac{3}{4} = \frac{21}{20}$

and $[1;20] = 1 + \frac{1}{20} = \frac{21}{20}$

Theorem 3

Let $a = \alpha_0$ be an irrational number and define the sequence $a_0, a_1, a_2, \ldots$ recursively by $a_k = \alpha_k$, $\alpha_{k+1} = \dfrac{1}{\alpha_k - a_k}$ for k =0,1,2,… . Then α is the value of infinite S.C.F. $[a_0; a_1, a_2, \ldots]$. For example $\sqrt{3} = [a_0; a_1, a_2, a_3, a_4, a_5, \ldots] = [1;1,2,1,2\ldots] = [1;\overline{1,2}]$. We can use the same operations of finite S.C.F. for infinite S.C.F..

Example 10

Find $[1;\overline{1,2}] \times [2;\overline{4}]$

Solution:

Let $[1;\overline{1,2}] = [1;1,2,1,2,\ldots] = [a_0; a_1, a_2, a_3, a_4, \ldots]$

and $[2;\overline{4}] = [2;4,4,4,4,\ldots] = [b_0; b_1, b_2, b_3, b_4, \ldots]$

from (2a) we have

$[a_0; a_1, a_2, a_3, a_4, \ldots] \times [b_0; b_1, b_2, b_3, b_4, \ldots] = [d_0; d_1, d_2, d_3, d_4, \ldots]$

where

$$d_0 = a_0 b_0 = 1 \cdot 2 = 2$$

$$d_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + 1} \right\rfloor = \left\lfloor \frac{1 \cdot 4}{1 \cdot 1 + 2 \cdot 4 + 1} \right\rfloor = \left\lfloor \frac{4}{10} \right\rfloor = 0$$

$$d_2 = \left\lfloor \frac{a_0 b_2 (a_1 a_2 + 1) + a_2 b_0 (b_1 b_2 + 1) + a_2 b_2}{(a_1 a_2 + 1)(b_1 b_2 + 1) - d_1 [a_0 b_2 (a_1 a_2 + 1) + a_2 b_0 (b_1 b_2 + 1) + a_2 b_2]} \right\rfloor$$
$$= \left\lfloor \frac{1(4)(3) + 2(2)(17) + 2(4)}{(3)(17)} \right\rfloor$$
$$= \left\lfloor \frac{88}{51} \right\rfloor = 1$$

$$d_3 = \left\lfloor \frac{K_3(a_1)K_3(b_1)}{[a_0K_3(a_1)K_2(b_2) + b_0K_2(a_2)K_3(b_1) + K_2(a_2)K_2(b_2)] - K_3(a_1)K_3(b_1)d_2} \right\rfloor$$

$$= \left\lfloor \frac{4(72)}{[4(17) + 2(3)(72) + 3(17)] - 4(72)} \right\rfloor$$

$$= \left\lfloor \frac{288}{263} \right\rfloor = 1$$

$$d_4 = \left\lfloor \frac{[a_0K_3(b_2)K_4(a_1) + b_0K_3(a_2)K_4(b_1) + K_3(a_2)K_3(b_2)] - K_4(a_1)K_4(b_1)}{K_4(a_1)K_4(b_1)K_2(d_2) - [a_0K_3(b_2)K_4(a_1) + b_0K_3(a_2)K_4(b_1) + K_3(a_2)K_3(b_2)]} \right\rfloor$$

$$= \left\lfloor \frac{[72(11) + 2(8)(305) + 8(72)] - 11(305)}{(3355)(2) - (6248)} \right\rfloor$$

$$= \left\lfloor \frac{2893}{462} \right\rfloor = 6$$

therefore

$[1;\overline{1,2}] \times [2;\overline{4}] = [2;0,1,1,6,\ldots]$

$= [3;1,6,\ldots]$ (Lemma 2)

To check, we have: $[1;\overline{1,2}] \times [2;\overline{4}] = \sqrt{3} \times \sqrt{5} = \sqrt{15}$

and $[1,\overline{1,2}] \times [2,\overline{4}] = [3,1,6,\ldots] = \sqrt{15}$

Conclusion

This paper is the Second part for the operations of the simple continued fractions. In the first part [6] we discovered the definitions of addition and subtractions of simple continued fractions. In this part we defined the multiplication, multiplicative inverse and the division of the simple continued fractions.

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