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APPLICATION OF COMPLEX NUMBERS IN PLANIMETRY

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Summary: Complex unit module numbers assigned to the vertices of a polygon have the greatest practical application in certain planimetric problems. Planimetry is the part of geometry that studies sets of points in the Euclidean plane. Some special sets of points, such as length, angle, circle, and circle, are basic planimetric elements. From these basic elements, more complex ones are formed, such as geometric figures in general (characters), especially polygonal lines and polygons. In addition to studying the properties of planimetric elements and their mutual relations, methods of constructing complex elements from basic, so-called geometric constructions.

Keywords: planimetry, complex numbers, unit circles, computational operations

1. INTRODUCTION

The realization of the idea of developing the concept of number permeates the entire methodological and content line of the school mathematics course, appearing to a greater or lesser extent in all grades of high school. This fully corresponds to the role of number as a fundamental concept in modern mathematics. Number is the most important means by which a person realizes the quantitative relations of the real world. In the process of studying various number systems, students actively develop their mathematical development, enrich their ideas, not only about the internal laws of development of mathematical ideas, but also about the connection of mathematics with practical needs, form and improve their arithmetic culture. In fact, a sufficiently good knowledge of the properties of numerical sets can enable successful mastery, as in the entire school course "Algebra and the beginnings of analysis", but also of each of its topics.

By introducing the root operation, we realized that the sets of numbers we had learned by then were not enough, so we became acquainted with the numbers we called irrational numbers and marked the set of these numbers with the letter I. The set was called rational and irrational numbers together with the letter R. However, the set of real numbers also had to expand due to the impossibility of solving some equations in it. The set of real numbers has been extended to a set of complex numbers. The set of complex numbers can be interpreted as the Euclidean plane, that is, starting from the theory of complex numbers, we can construct a model of the Euclidean plane[1],[2]. Mappings of a set of complex numbers into a set of complex numbers are then interpreted as transformations of the Euclidean plane. It turns out that it is extremely convenient to work that way. Namely, isometric transformations (translation, rotation, axial reflection), similarity transformations (homothety) and inversions are given by much simpler formulas. Also, many important theorems such as: On the center of gravity of a triangle, On the center of the circumcircle of a triangle, On the orthocenter of a triangle, On the Euler line are simply expressed and proved by complex numbers[3],[4].

2. SET OF COMPLEX NUMBERS

The set of complex numbers can be interpreted as a Euclidean plane, that is, starting from the theory of complex numbers, we can construct a model of the Euclidean plane. Mappings of a set of complex numbers into a set of complex numbers are then interpreted as transformations of the Euclidean plane. It turns out that it is extremely convenient to work that way. Namely, isometric transformations (translation, rotation, axial reflection), similarity transformations (homothety) and inversions are given by much simpler formulas. Also, many important theorems, such as: on the center of gravity of a triangle, on the center of the circumcircle of a triangle, On the orthocenter of a triangle, on Euler's line, are simply expressed and proved using complex numbers. It is also interesting to prove the position that claims that by inverting a generalized circle, they are painted into generalized circles. In addition to the study of elementary geometry, complex numbers are extremely suitable for the study of hyperbolic geometry (for example, the Poincare disk model and the Poincare semi-plane model).

As already mentioned, the set of complex numbers was created by expanding the set of real numbers[5]. Although from today's perspective it seems to us that complex numbers were introduced for the purpose of solving a quadratic equation, this is not true. At the time complex numbers were discovered, the quadratic equation had been known for more than 3,000 years, and to solve a quadratic equation it was enough to know that it could have two, one, or no solutions. The reason for the discovery of complex numbers was to solve the cubic equation. Complex numbers could not be bypassed here[6]. General algebraic equation of the third degree

$$ax^3 + bx^2 + cx + d = 0$$

decided Scipio del Ferro, who was a student and professor at the University of Bologna. Independently of him, Nicolo Fontana Tartaglia also managed to solve the cubic equation, keeping his method of solving a secret until 1539, when he revealed the secret to the Italian mathematician Girolamo Cardano. A couple of years later, Cardano published a work called Ars magna (Great Skill) in which he published formulas for solving the cubic equation, which is why the formulas for solving such an equation are still called Cardano's formulas[7]. Complex numbers in mathematics have been brought into equal position with real numbers, with Abraham de Moivre and Leonhard Euler being particularly credited. Later, complex numbers were connected with geometry, in which the mathematician Carl Friedrich Gauss had a special merit.

3. PROPERTIES OF THE UNIT CIRCLE

Complex numbers of a unit module assigned to the vertices of a polygon have the greatest practical application in specific planimetric problems[8]. Then many seemingly unfeasible computational operations, both temporally and technically, became very easy and elegantly performed[9],[10].

3.1. Tendon of a unit circle

Theorem 3.1. *For points a and b the unit of a circle holds*

$$\frac{a-b}{\overline{a}-\overline{b}} = -ab.$$

Proof. The proof is trivial and follows, after a short calculation, using the fact that it is

$$\overline{a} = \frac{1}{a} i \ \overline{b} = \frac{1}{b}.$$

Theorem 3.2. *If the point c on the chord ab of the unit circle then holds*

$$\overline{a} = \frac{a+b-c}{ab}.$$

Proof. Since the points a, b and c are collinear, hence by the theorem we have that for the lines determined by points a and b and points c and a

$$\frac{a-b}{a-c} = \overline{\left(\frac{a-b}{a-c}\right)}$$

whence using the facts $\overline{a} = \frac{1}{a} i \ \overline{b} = \frac{1}{b}$, as in the previous proof, after a short account, we get our claims.

Theorem 3.3. *The base of an arbitrary point c on the chord ab of a unit circle is a point*

$$p = \frac{1}{2}\left(a+b+c-ab\overline{c}\right).$$

Proof. As p is the base of the normal from c to the tendon ab we have that

$$\frac{c-p}{\overline{c}-\overline{p}} = -\frac{a-b}{\overline{a}-\overline{b}}.$$

Using the condition | a | = | b | = 1 we get that the last equation is equivalent to

$$\frac{c-p}{\overline{c}-\overline{p}} = ab \Leftrightarrow \overline{p} = \frac{ab\overline{c}+p-c}{ab}.$$

As p belongs to the chord ab of a unit circle, we have by the theorem that

$$\overline{p} = \frac{a+b-p}{ab}.$$

Equating the last two equations, we get that

$$\frac{ab\overline{c}+p-c}{ab} = \frac{a+b-p}{ab} \Leftrightarrow p = \frac{1}{2}\left(a+b+c-ab\overline{c}\right)$$

Theorem 3.4. *The intersection of the chords ab and cd of a unit circle is a point*

$$t = \frac{ab(c+d)-cd(a+b)}{ab-cd}.$$

Proof. The point t as the cross section of the tendon belongs to both tendons. Then, based on the theorem, we have that

$$\overline{t} = \frac{a+b-t}{ab}.$$

Symmetrically we have that

$$\overline{t} = \frac{c+d-t}{cd}.$$

Equating the right sides of the last two equations and solving by t, as in the previous proof, we get the statement of the theorem after a short calculation.

3.1. Tangent of a unit circle

The following theorem gives us the coordinate of a point obtained as the intersection of two tangents of a unit circle, if that intersection exists at all[11].

Theorem 3.5. *The intersection of the tangents at points a and b, a ≠ - b, the unit circle is the point*

$$\frac{2ab}{a+b}.$$

Proof. Let us denote the intersection of the tangents by p. Let the center of the circle be placed in the coordinate origin without diminishing the generality. We have it

$$ra \perp a0 \iff \frac{r - a}{\bar{r} - \bar{a}} = -a^2 \iff \bar{r} = \frac{2a - r}{a^2}.$$

As the problem is symmetric by the letters a and b in relation to the coordinate origin and the point r we have that

$$\bar{r} = \frac{2b - r}{b^2}.$$

From the last two equations, by equating and solving by r, we get that

$$r = \frac{2ab}{a+b}.$$

3.1.1 Equation of the tangent of a unit circle

Let the arbitrary point t be on the tangent at the point x of the unit circle. Then the line tx, given the classical geometric definition of the tangent to the circle, is normal to the radius xo. Using the condition of normality from the theorem, we have to hold that it is

$$\frac{t - x}{\bar{t} - \bar{x}} = -\frac{x - o}{\bar{x} - \bar{o}}.$$

Since o = 0, we get that the equation of the tangent at an arbitrary point x is a unit circle, using the condition $x\bar{x} = 1$.

$$t = x(2 - x\bar{t}).$$

3.2. A unit circle inscribed in a triangle

When an inscribed circle of a triangle occurs in problems, in many situations it is useful, and in some situations the only possible solution, that the center of the inscribed circle coincides with the coordinate origin and the inscribed circle is unit.

Theorem 3.6. *Let the unit circle be inscribed in the triangle abc and let it touch the sides bc, ca and ab at the points p, q and r respectively. Then it is valid*

$$a = \frac{2qr}{q+r}, \quad b = \frac{2qr}{q+r}, \quad c = \frac{2pq}{p+q}.$$

Proof. The proof follows directly by applying the theorem

Theorem 3.7. *Let the unit circle be inscribed in the triangle abc, and let it touch the sides bc, ca and ab at the points p, q and r respectively. Then the orthocenter is determined by a point*

$$h = \frac{2(p^2q^2 + q^2r^2 + r^2p^2 + pqr + (p+q+r))}{(p+q)(q+r)(r+p)}.$$

Proof. Let the coordinate origin be at the point of the center of the circumscribed circle. Hence we have, by the theorem, that for o = 0 we have x = a + b + c; which is by Theorem 1:6

$$h = \frac{2qr}{q+r} + \frac{2rp}{r+p} + \frac{2pq}{p+q}.$$

whence, by bringing it to a common denominator, after a short calculus, we obtain the assertions of the theorem.

Theorem 3.8. *Let the unit circle be inscribed in the triangle abc and let it touch the sides bc, ca and ab at the points p, q and r respectively. Then for the center of the circumcircle of the triangle abc holds*

$$o = \frac{2pqr+(p+q+r)}{(p+q)(q+r)(r+p)}.$$

Proof. By theorem we have to be valid

$$h + 2o = a + b + c.$$

From there we have that

$$o = \frac{a+b+c-h}{2}.$$

Using Theorems 1.6 and 1.7 we have valid

$$o = \left(\frac{1}{2}\ \frac{2qr}{q+r} + \frac{2rp}{r+p} + \frac{2pq}{p+q} - \frac{2(p^2q^2+q^2r^2+r^2p^2+pqr+(p+q+r))}{(p+q)(q+r)(r+p)}\right)$$

from where, by bringing to the common denominator, the statement of the theorem is obtained with a shorter calculus.

3.4. A triangle inscribed in a unit circle

In practical problems, more than one circle often occurs. In many situations where an inscribed and attributive circle of a triangle occurs in combination with a described circle of a triangle, it is useful, and sometimes the only possible way to solve the problem, by using the following statements.

Theorem 3.9. *For each triangle abc inscribed in a unit circle, there are numbers u, υ and ω such that* $a = u^2$, $b = \upsilon^2$ *ic* $= \omega^2$, *and the centers of the arcs ab, bc and ca that do not contain points c, a and respectively, are points* $-u\upsilon$, $-\upsilon\omega$ *and* $-\omega u$ *respectively.*

Proof. The first part of the statement of the theorem is trivial. We will prove only the second part of the claim. Let us first emphasize that │a│ = │b│ = │c│ = │u│ = │ υ │ = │ ω │ = 1. Let us now prove the following auxiliary statement:

Lemma 1. A line from an arbitrary vertex of a triangle, which also contains the center of an inscribed circle, is a pole of an arc determined by the other two vertices, which does not pass through the first vertex.

Proof of Lemma 1. Let a line, without losing generality, be constructed from the vertex a = u2. Let the intersection point of that right and described circle be the point x, where │x│ = 1. As ∠cox = 2∠cax, as the central and peripheral angle over the chord cx. Symmetrically we have that ∠xob = 2∠xab, as the central and peripheral angle over the tendon bx. Hence we have that ∠cox = ∠xob, because ∠cax = ∠xab. Hence we have that if the central angles over the tendons are equal, then the tendons are also equal, so are the arcs over those tendons, whence we have our auxiliary assertion.

Let it be

$$a = e^{i(2a)},\ b = e^{i(2\beta)},\ c = e^{i(2\gamma)},\ 0 \le \alpha < \beta < \gamma \le \pi.$$

Choose the numbers *u*, υ and ω as:

$$u = e^{i\alpha}, \quad v = e^{i(\beta+\pi)}, \quad \omega = e^{i\gamma}.$$

From here we have it in order

$$-uv = e^{i(\alpha+\beta)}, \quad -v\omega = e^{i(\beta+\gamma)}, \quad -\omega u = -e^{i(\alpha+$$

Since the angles corresponding to the centers of the arcs ab, bc and ca that do not contain the points c, a and b are respectively the angles α + β, β + γ, α + γ

+ π and the previous equations, we have the statement of the theorem.

Theorem 3.10. *In addition to the previously described conditions in theorem 3.9 for the center of the inscribed circle and is valid*

$$i = -(uv + v\omega + \omega u).$$

Proof. Let us first prove the following auxiliary statement:

Lemma 2. Given a triangle with properties from Theorem 3.8. Then the ortho-center of the triangle obtained by construction is longer between the points which are the centers of the arcs coinciding with the center of the inscribed circle of the triangle abc.

Proof of Lemma 2. Let them be in order -υω = x, -υω = y, -uυ = z, ∠cax = ∠xaz = α, ∠aby = ∠ybc = β and ∠acz = ∠zcb = γ. Then α + β + γ = π / 2. Denote by p the intersection of the line ax and the sides yz of the triangle xyz. Since ∠yxa = β and ∠xyz = α + γ, we have that ∠ypx = π / 2. We get analogous for the other two bases. From there we have that the point and orthocenter of the triangle xyz, i.e. triangle determined by the points - uυ, -υω, i-ωu, which proved the lemma.
Using the previous lemma, we obtain, with the assertion of Theorem 2: 8 and o ≡ 0, that the orthocenter of a triangle is determined by

$$h \equiv i = x + y + z = -uv - v\omega - \omega u = -(uv + v\omega + \omega u)$$

thereby proving the theorem.

Definition 6. An ascribed circle of a triangle is a circle that externally touches the side of the triangle and the extensions of the other two sides of that triangle.

Theorem 3.11. *The center of the assigned circle of the side ab of the triangle abc is determined by the expression*

$$j = -uv + v\omega + \omega u$$

where the numbers u, v and! numbers from Theorem 3:9.

Proof. The proof follows trivially from the orthogonality of the lines ib and the bisector of the exterior angle of the triangle abc at the vertex b and the line ia and the bisector of the exterior angle of the triangle abc at the vertex a.

4. CONCLUSION

Complex numbers have found application in various spheres of mathematics. After a historical review and introduction of basic definitions, a presentation of various and late application of complex numbers is given. The aim of the paper was to show how complex numbers provide the possibility to solve certain mathematical problems solves more efficiently and elegantly which is applied in solving problems in algebra, analysis and analytical geometry. Basic geometric concepts in analytical geometry can be easily expressed through complex numbers, and it is possible to solve various geometric problems

with the use of complex numbers. They can also be a powerful apparatus in algebra for determining finite sums and sums, sums generated by a binomial pattern. We can use complex numbers to calculate finite products, in polynomial algebra, as well as in solving trigonometric equations. In the analysis, complex numbers were found application in solving several types of indefinite integrals. On the other hand, the application of complex numbers simplifies and in some way makes the solutions of some problems more "elegant", but their use is not necessary because all problems can be solved without them.

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