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Original Article- ***Mathematics***

Derivatives and its relation with continuity

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ABSTRACT:
Derivative is an important part in mathematics. The derivative of a function y = f(x) with respect to x ix denoted by dy/dx or y' or f'(x) etc. The derivative dy/dx has two aspects (Interpretations): One is the geometrical and next physical. In geometrical aspect dy/dx measures the slope of the curve y = f(x) at a point (x, y) and in physical aspect dy/dx measures the rate of change of y = f(x) with respect to x.
Keywords: Derivative, left and right derivative, Continuity, interval, interior point, end point, limit, function

INTRODUCTION:

In mathematics, Darivative and continuity have close relation. Continuity of a function at a point means the graph of the function at that point has no break i.e the graph runs over the point. Existence of derivative of a function at a point means tangent can be drawn on the curve over the point. The slope of the tangent at a point (c, f (c)) gives the derivative of the function at the pont c.

i- Some definitions.

a- Differentiabilty at an interior point.

Let f be defined on an interval I in R and let c is an interior point of I. we say f is differentiable (or derivable) at the point c, if the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ $\lim_{h\to 0}$ or $\frac{f(c+h)-f(c)}{h}$ (h>0) exisists (finitely).

This limit (if it exists) is called the derivative (or differential coefficient) of f at x=c and it is denoted by $f'(c)$.

Thus, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ (if exists) = $f'(c)$, is called the derivative of f at c. The process of evaluating $f'(c)$ is called differentiation.

Notes:(1) The symbols Df(x), Df, f', f^1, f_1, f_x, or $\frac{df}{dx}$ are also used for the derivative $f'(x)$

(2) Graphically, $f'(c)$ means the slope (gradient), of the curve y=f(x) at the points (c, f(c)) and quantitatively, $f'(c)$ means the rate of change of the function at c w.r.t. x.

Remarks: The following 3 statements are equivalent –

1- The derivative of f(x) at x=c i.e. $f'(c)$ exists.

2- The limit, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f^1(c)$ exists.

3- $\forall \in > 0\ \exists \delta > 0$ s.t. $|x-c| < \delta \Rightarrow \left| \frac{f(x)-f(c)}{x-c} - f^1(c) \right| < \in$.

An example: Let f:R$\to$R is defined by f(x) = x^2 $\forall x \in R$.

Then at any point $c \in R$, we have-

$$f^1(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{x^2-c^2}{x-c} = \lim_{x\to c} (x+c) = 2c$$

$\Rightarrow$ f' exists at arbitrary point $c \in R$.

$\Rightarrow$ f' exists on R and $f'(x)=2x$.

b- One sided derivatives.

Let f be defined on an interval I=(a, b) in R and let $c \in I$. Then we have-

i-Left hand derivative: f is said to be differentiable from the left at c, if the left hand limit-

$\lim_{x \to c-} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0-} \frac{f(c+h)-f(c)}{h} (h<0)$ or $\lim_{h \to 0} \frac{f(c+h)-f(c)}{-h}$ (h>0) exists.

This left hand limit (if it exists) is called the left hand derivative of f at x =c and it is defined by $f'(c-)$ or $Lf'(c)$.

Thus, $\lim_{x \to c-} \frac{f(x)-f(c)}{x-c}$ (if exists) = $f'(c-)$, is called the left hand derivative of f at x=c.

ii- Right hand derivative: f is said to be differentiable from the right at c, if the right hand limit-

$\lim_{x \to c+} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0+} \frac{f(c+h)-f(c)}{h}$ (h>0) or $\lim_{h \to 0+} \frac{f(c+h)-f(c)}{h}$ (h>0) exists.

This right hand limit (if it exists) is called the right hand derivative of f at x=c and it is defined by $f'(c+)$ or R $f'(c)$.

Thus, $\lim_{x \to c+} \frac{f(x)-f(c)}{x-c}$ (if exists)= $f'(c+)$, is called the right hand derivative of f at x=c.

Notes:- (1) $f'(c)$ exists $\Leftrightarrow$ $f'(c-)$ and $f'(c+)$ exist and are equal. In fact, $f'(c) = D \Leftrightarrow f'(c-) = f'(c+) = D$.

(2) The left hand derivative $f'(c-)$ is also called the regressive derivative and the right hand derivative $f'(c+)$ is also called the progressive derivative.

c- Derivative at the end points:-

Let f be a function defined on a closed interval [a, b], then-

i- f is said to be differentiable at the left end point 'a' if the right derivative at 'a' i.e. $f'(a+)$ exists, and we write-
$f'(a) = f'(a+)$.

ii- f is said to be differentiable at the right end point 'b' if the left derivative at 'b' i.e. $f'(b-)$ exists, and we write-
$f'(b) = f'(b-)$.

d- Differentiability on an interval.

Let f be a function defined on a closed interval [a, b]. Then-

i- f is said to be derivable on the open interval (a, b) if f is derivable at all point of (a, b).

i.e. if $f'(c)$exists $\forall c \in (a, b)$.

ii- f is said to be derivable on the closed interval [a, b] if f is derivable at all points of [a, b].

i.e. if $f'(a+)$, $f'(b-)$ exist and $\forall c \in (a, b)$ $f'(c)$ exists.

Obiviously, $f'(c)$ exists $\Leftrightarrow$ $f^1(c-)$, $f^1(c+)$ exists and are equal.

Notes: (1) Definitions similar to the above can be given for the derivability of f on the intervals-
[a, b), (a, b], $(-\infty, a), (-\infty, a], (a, \infty), [a, \infty)$.

(2) $\frac{df}{dx} = f'$ is also a function, and if the domains of f and f' are D and D' respectively, then $D' \subseteq D$.

Obiviously, $D' = \{x \in D : f'(x) \text{ exists}\}$.

Some examples:

Ex.1- Show that $f(x) = x^2$ is derivable on [0, 1].

Solutation: Clearly, $\forall c \in (0,1)$, we have $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \frac{x^2-0}{x-0} = 2c$.

At x= 0, $f'(0) = f'(0+) = \lim_{x \to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0+} \frac{x^2-0}{x-0} = \lim_{x \to 0+} x = 0$.

And at x =1, $f'(1) = f'(1-) = \lim_{x \to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1-} \frac{x^2-1}{x-1} = \lim_{x \to 1-} (x+1) = 2$

Thus we see that f' exists on [0,1] i.e. f is derivable on [0,1].

Ex.2: Let $f(x) = 2|x| + |x-2|$, find $f'(1)$.

Solution: we have $|x| = x$, when x>0 (so at x=1)

and $|x-2| = -(x-2) = 2-x$ when x<2 (so at x= 1)

$\therefore$ f(x) = 2x + 2 – x = x + 2 $\forall x \in (0,2)$

Now, $f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{(x+2)-(1+2)}{x-1} = \lim_{x\to 1} \frac{x-1}{x-1} = 1$. Ans.

Ex.3. Let $f(x) = \begin{cases} x & if\ 0 \le x < 1 \\ 1 & if\ \ x \ge 1 \end{cases}$

Show that f'(1) doesnot exist.

Soln: $Lf'(1) = f'(1-) = f'(1-) = \lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x-1}{x-1} = 1$, and-

$Rf'(1) = f'(1+) = \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+} \frac{1-1}{x-1} = 0$

Since L f'(1) $\neq$ R f'(1), so f'(1) doesnot exists.

Ex.4- let $f(x)=x^2 \sin \frac{1}{x}$ if $x \neq 0$

0 if x = 0

Show that $f^1(0)$ exists but $\lim_{x\to 0} f^1(x) \neq f^1(0)$.

Soln: $f^1(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x\to 0} (x \sin \frac{1}{x}) = 0$ exists.

Now, from elementary calculus, we have $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ $\forall x \neq 0$.

Clearly, the limit $\lim_{x\to 0} f'(x)$ does not exist and therefore there is no possibility of $\lim_{x\to 0} f'(x)$ to be equal to $f^1(0)$. Which indicates that the function f' is not continuous at the point x = 0 though $f^1(0)$ exists.

ii- Derivability and continuity

Theorem: if a function f is derivable at a point then it is continuous at that point. But the converse may not be true. In other words, The continuity of a function at a point is a necessary but not a sufficient cond^n for the existence of the derivative of the function at that point.

Proof

Here we have to show:

i- Derivability of f at a point $\Rightarrow$ Continuity of f at the point.

ii- Continuity of f at a point not $\Rightarrow$ Derivability of f at the point.

To show (i): Let a function f is derivable at the point c then the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists and equal to f'(c).

i.e. $f^1(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ (x $\neq$ c), exists.

Now, $f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \times (x-c)$ ($\because$ x-c $\neq$ 0)

$\Rightarrow \lim_{x\to c} [f(x)-f(c)] = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \times \lim_{x\to c} (x-c)$

$= f'(c) \times 0$

$=0$

$\Rightarrow \lim_{x\to c} f(x) - f(c) = 0$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\Rightarrow$ f is continuty at the point x =c.
Thus we showed (i).

To show ii. Consider the function f:R $\longrightarrow$ R, defined by f(x) = $|x| \; \forall x \in R$.
Then f is continuty at the point x=0 but f'(0) does not exist.

<u>To show f is continuty at x =0</u>

$\lim_{x \to 0-} f(0-) = \lim_{x \to 0-} f(x) = |x| = (-x) = 0$, and

$\lim_{x \to 0+} f(0+) = \lim_{x \to 0+} f(x) = \lim_{x \to 0+} |x| = (x) = 0$

And f(0) = $|0|$ =0.

Since f(0-) = f(0+) = f(0), so f is continuous at x=0.

<u>To show f'(0) does not exist</u>

$$\lim_{x \to 0-} f'(0-) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x| - |0|}{x - 0} = \frac{-x}{x} = -1, \quad \text{and}$$

$$f'(0+) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0+} \frac{x}{x} = 1$$

Since, f'(0-) $\neq$ f'(0+), so f'(0) does not exist i.e. f is not derivable at x=0.
Hence the theorem

Remarks: the function f(x)= $|x|$ is derivable at every $x \in R$ except x =0.

In fact: $f'(x) = \begin{cases} 1 \; \forall x > 0 \\ -1 \; \forall x < 0 \end{cases}$

Concluding Remark: A function f is derivable(Differentiable) at a point c i.e. a tangent can be drawn at the point (c, f (c)) on the graph of f only when the f is continuous at the point c. The converse may not be true i.e. continuity of f at a point c may not exist derivative at that point. Hence, continuity of f at c is the necessary for f to be differentiable at c but not sufficient.

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